

Find the general solution of $y'' - 5y' - 6y = xe^{-x}$.

SCORE: ____ / 14 PTS

$$r^2 - 5r - 6 = 0$$
$$(r - 6)(r + 1) = 0$$

UNDETERMINED COEFFICIENTS
SOLUTION

$$r = 6, -1$$

$$y_h = \underline{c_1 e^{6x} + c_2 e^{-x}} \quad \textcircled{2}$$

$$y_p = x(Ax + B)e^{-x} = \underline{(Ax^2 + Bx)e^{-x}} \quad \textcircled{2}$$

$$y_p' = (2Ax + B)e^{-x} + (-Ax^2 - Bx)e^{-x} = \underline{(-Ax^2 + (2A - B)x + B)e^{-x}} \quad \textcircled{2}$$

$$y_p'' = (-2Ax + (2A - B))e^{-x} + (Ax^2 + (-2A + B)x - B)e^{-x}$$

$$y_p'' = \underline{(Ax^2 + (-4A + B)x + (2A - 2B))e^{-x}} \quad \textcircled{2}$$

$$-5y_p' + (5A^2 + (-10A + 5B)x - 5B)e^{-x}$$

$$-6y_p + (-6Ax^2 - 6Bx)e^{-x}$$

$$= \underline{\textcircled{2} (-14Ax + (2A - 7B))e^{-x}} = xe^{-x}$$

$$-14A = 1 \rightarrow A = -\frac{1}{14} \quad \textcircled{1}$$

$$2A - 7B = 0 \rightarrow B = \frac{2}{7}A = -\frac{1}{49} \quad \textcircled{1}$$

$$y = \underline{\textcircled{1} \left(-\frac{1}{14}x^2 - \frac{1}{49}x\right)e^{-x}} + \underline{c_1 e^{6x} + c_2 e^{-x}} \quad \textcircled{1}$$

OR

VARIATION OF PARAMETERS
SOLUTION

$$y_1 = e^{-x} \quad y_2 = e^{6x} \quad (2)$$

$$W = \begin{vmatrix} e^{-x} & e^{6x} \\ -e^{-x} & 6e^{6x} \end{vmatrix} = 7e^{5x} \quad (2)$$

$$y_p = -e^{-x} \int \frac{x e^{-x} \cdot e^{6x}}{7e^{5x}} dx + e^{6x} \int \frac{x e^{-x} \cdot e^{-x}}{7e^{5x}} dx$$

$$= \underbrace{-\frac{1}{7}e^{-x} \int x dx}_{(2)} + \underbrace{\frac{1}{7}e^{6x} \int x e^{-7x} dx}_{(2)}$$

$$= -\frac{1}{7}e^{-x} \left(\frac{1}{2}x^2 \right) + \frac{1}{7}e^{6x} \left(-\frac{1}{7}x e^{-7x} - \frac{1}{49}e^{-7x} \right) \quad \begin{matrix} u & dv \\ x & e^{-7x} \end{matrix}$$

$$= -\frac{1}{14}x^2 e^{-x} - \frac{1}{49}x e^{-x} + C e^{-x} \quad (2)$$

$\begin{matrix} 1 & + & -\frac{1}{7}e^{-7x} \\ 0 & - & \frac{1}{49}e^{-7x} \end{matrix}$

$$y = \underbrace{-\frac{1}{14}x^2 e^{-x} - \frac{1}{49}x e^{-x}}_{(2)} + \underbrace{C_1 e^{6x} + C_2 e^{-x}}_{(1)}$$

Use the annihilator method to find the form of a particular solution of $y'' + 4y' + 5y = 6x^2 - 3e^{-2x} \sin x$.

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You MUST show the usage of the annihilator. Do NOT solve for the coefficients in the particular solution.

D^3 ANNihilATES $6x^2$

$$r^2 + 4r + 5 = 0$$

$(D+2)^2 + 1$ ANNihilATES $-3e^{-2x} \sin x$

$$r = -2 \pm i$$

$$y_h = C_1 e^{-2x} \cos x + C_2 e^{-2x} \sin x$$

$$D^3 ((D+2)^2 + 1) (D^2 + 4D + 5) [y] = D^3 ((D+2)^2 + 1) [6x^2 - 3e^{-2x} \sin x]$$

$$D^3 ((D+2)^2 + 1)^2 [y] = 0 \rightarrow r = 0, 0, 0, -2 \pm i, -2 \pm i$$

$$y = \underbrace{A + Bx + Cx^2}_{\frac{1}{2}} + \underbrace{De^{-2x} \cos x + Ee^{-2x} \sin x}_{y_h, \frac{1}{2}} + \underbrace{Fxe^{-2x} \cos x + Gxe^{-2x} \sin x}_{\frac{1}{2}}$$

$$y_p = \underbrace{A + Bx + Cx^2 + Fxe^{-2x} \cos x + Gxe^{-2x} \sin x}_1$$

Find the general solution of $x^2 y'' + xy' + 4y = \csc(2 \ln x)$. $\rightarrow y'' + \frac{1}{x} y' + \frac{4}{x^2} y = \frac{\csc(2 \ln x)}{x^2} = g(x)$ SCORE: ____ / 10 PTS

$$r^2 + 4 = 0$$

$$r = \pm 2i$$

$$\textcircled{2} y_1 = \cos(2 \ln x), y_2 = \sin(2 \ln x)$$

$$W = \begin{vmatrix} \cos(2 \ln x) & \sin(2 \ln x) \\ -\frac{2}{x} \sin(2 \ln x) & \frac{2}{x} \cos(2 \ln x) \end{vmatrix} = \frac{2}{x} \textcircled{1}$$

$$y_p = -\cos(2 \ln x) \int \frac{\frac{\csc(2 \ln x)}{x^2} \sin(2 \ln x)}{\frac{2}{x}} dx$$

$$+ \sin(2 \ln x) \int \frac{\frac{\csc(2 \ln x)}{x^2} \cos(2 \ln x)}{\frac{2}{x}} dx$$

$$= \textcircled{1} -\cos(2 \ln x) \int \frac{1}{2x} dx + \sin(2 \ln x) \int \frac{\cos(2 \ln x)}{2x \sin(2 \ln x)} dx \textcircled{1\frac{1}{2}}$$

$$= -\cos(2 \ln x) \left(\frac{1}{2} \ln |x| \right) + \sin(2 \ln x) \left(\frac{1}{4} \ln |\sin(2 \ln x)| \right) \textcircled{1}$$

$$y = \textcircled{2} \frac{1}{2} \cos(2 \ln x) \ln |x| + \frac{1}{4} \sin(2 \ln x) \ln |\sin(2 \ln x)|$$

$$+ C_1 \cos(2 \ln x) + C_2 \sin(2 \ln x) \textcircled{1}$$

BONUS QUESTION:

Find a second order linear differential equation whose general solution is

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$$y = \underbrace{Ax^2 + Bx^2 \ln x}_{y_h} + \underbrace{x^3}_{y_p}$$

CAUCHY-EULER y_h

$$r = 2, 2$$

$$(r-2)^2 = 0$$

$$r^2 - 4r + 4 = 0$$

$$\textcircled{1} \quad x^2 y'' - 3xy' + 4y = b(x)$$

$$\textcircled{1} \quad x^2 (6x) - 3x (3x^2) + 4(x^3) = x^3$$

$$\textcircled{1} \quad x^2 y'' - 3xy' + 4y = x^3$$